## Even and Odd Functions

A function, $f$, is *even* (or *symmetric*) when

$$f(x) = f(-x).$$

A function, $f$, is *odd* (or *antisymmetric*) when

$$f(x) = -f(-x).$$

**Theorem 5.1** Any function can be written as a sum of even and odd functions.

$$\begin{aligned} f(t) &= \frac{1}{2}\left[f(t)+\underbrace{f(-t)-f(-t)}_{0}+f(t)\right] \\ &= \underbrace{\frac{1}{2}[f(t)+f(-t)]}_{f_e}+\underbrace{\frac{1}{2}[f(t)-f(-t)]}_{f_o} \end{aligned}$$

$f_e$ is even because $f_e(t) = f_e(-t)$:

$$\begin{aligned} f_e(t) &= f(t)+f(-t) \\ &= f(-t)+f(t) \\ &= f_e(-t). \end{aligned}$$

$f_o$ is odd because $f_o(t) = -f_o(-t)$:

$$\begin{aligned} f_o(t) &= f(t)-f(-t) \\ &= -[f(-t)-f(t)] \\ &= -f_o(-t). \end{aligned}$$

## Even and Odd Functions (contd.)

**Theorem 5.2** The integral of the product of odd and even functions is zero.

$$\int_{-\infty}^{\infty} f_e(x) f_o(x) dx =$$

$$\int_{-\infty}^{0} f_e(x) f_o(x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx.$$

Substituting $-x$ for $x$ and $-dx$ for $dx$ in the first term yields:

$$\int_{\infty}^{0} -f_e(-x) f_o(-x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} f_e(-x) f_o(-x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} \left[ f_e(-x) f_o(-x) + f_e(x) f_o(x) \right] dx.$$

Substituting $f_e(-x)$ for $f_e(x)$ and $-f_o(-x)$ for $f_o(x)$ yields:

$$\int_{0}^{\infty} \underbrace{\left[ f_e(-x) f_o(-x) - f_e(-x) f_o(-x) \right]}_{0} dx.$$

# Fourier Transform Symmetry

The Fourier transform of $f(t)$ is defined to be:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt.$$

This can be rewritten as follows:

$$F(s) = \int_{-\infty}^{\infty} f(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} f(t) \sin(2\pi st) dt.$$

Substituting $f_e(t) + f_o(t)$ for $f(t)$ yields:

$$F(s) = \int_{-\infty}^{\infty} f_e(t) \cos(2\pi st) dt + \int_{-\infty}^{\infty} f_o(t) \cos(2\pi st) dt -$$

$$j \int_{-\infty}^{\infty} f_e(t) \sin(2\pi st) dt - j \int_{-\infty}^{\infty} f_o(t) \sin(2\pi st) dt.$$

## Fourier Transform Symmetry (contd.)

However, the second and third terms are zero (Theorem 5.2):

$$F(s) = \int_{-\infty}^{\infty} f_e(t)\cos(2\pi st)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(2\pi st)dt.$$

It follows that:

$$F(s) = F_e(s) + F_o(s).$$

Even Functions

**Theorem 5.3** The Fourier transform of a real even function is real.

$$
\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}dt \\
&= \int_{-\infty}^{\infty} f(t)\left[\cos(2\pi st) - j\sin(2\pi st)\right]dt \\
&= \int_{-\infty}^{\infty} f(t)\cos(2\pi st)dt
\end{aligned}
$$

which is real.

## Odd Functions

**Theorem 5.4** The Fourier transform of a real odd function is imaginary.

$$
\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \\
&= \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi st) - j \sin(2\pi st)\right] dt \\
&= j \int_{-\infty}^{\infty} f(t) \sin(2\pi st) dt
\end{aligned}
$$

which is imaginary.

## Even Functions (contd.)

**Theorem 5.5** The Fourier transform of an even function is even.

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt$$

Substituting $f(-t)$ for $f(t)$ yields:

$$F(s) = \int_{t=-\infty}^{t=\infty} f(-t) e^{-j2\pi st} dt.$$

Substituting $u$ for $-t$ and $-du$ for $dt$ yields:

$$\begin{aligned} &= \int_{u=\infty}^{u=-\infty} -f(u) e^{-j2\pi s(-u)} du \\ &= \int_{-\infty}^{\infty} f(u) e^{-j2\pi(-s)u} du \\ &= F(-s). \end{aligned}$$

Odd Functions (contd.)

**Theorem 5.6** The Fourier transform of an odd function is odd.

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt$$

Substituting $-f(-t)$ for $f(t)$ yields:

$$F(s) = \int_{t=-\infty}^{t=\infty} -f(-t) e^{-j2\pi st} dt.$$

Substituting $u$ for $-t$ and $-du$ for $dt$ yields:

$$\begin{aligned} &= \int_{u=\infty}^{u=-\infty} f(u) e^{-j2\pi s(-u)} du \\ &= \int_{-\infty}^{\infty} -f(u) e^{-j2\pi(-s)u} du \\ &= -F(-s). \end{aligned}$$

## Fourier Transform Symmetry (contd.)

- The Fourier transform of the even part (of a real function) is real (Theorem 5.3):

$$\mathcal{F}\{f_e\}(s) = F_e(s) = \mathrm{Re}(F_e(s)).$$

- The Fourier transform of the even part is even (Theorem 5.5):

$$\mathcal{F}\{f_e\}(s) = F_e(s) = F_e(-s).$$

- The Fourier transform of the odd part (of a real function) is imaginary (Theorem 5.4):

$$\mathcal{F}\{f_o\}(s) = F_o(s) = \mathrm{Im}(F_o(s)).$$

- The Fourier transform of the odd part is odd (Theorem 5.6):

$$\mathcal{F}\{f_o\}(s) = F_o(s) = -F_o(-s).$$

## Hermitian Symmetry

We can summarize all four symmetries possessed by the Fourier transform of a real function as follows:

$$\begin{aligned} F(s) &= F_e(s) + F_o(s) \\ &= F_e(-s) - F_o(-s) \\ &= \overline{F_e(-s) + F_o(-s)} \\ &= \overline{F(-s)}. \end{aligned}$$

## Hermitian Symmetry (contd.)

This symmetry matches the symmetry of the functions which comprise the Fourier basis:

$$e^{j2\pi st} = \overline{e^{j2\pi s(-t)}}.$$