Fourier Spectroscopy and the Causality Principle

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The consequences of the causality principle on the relations between absorption
and dispersion mode signals obtained in a repetitive pulse-Fourier experiment are
investigated with particular emphasis on the distribution of information between
the two signals. Interpolation processes to obtain higher frequency resolution are
analyzed and the influence on the inherent sensitivity is described.

INTRODUCTION

It is well known that the real and imaginary parts of the transfer function $Y(f)$ of a
linear, time-invariant physical system are related by a Hilbert transformation (*1–3*):

$$Y(f) = V(f) + iU(f),$$ [1]

$$U(f) = (-1/\pi f) * V(f) = -(1/\pi) \int_{-\infty}^{\infty} V(f')(f-f')^{-1} df',$$

$$V(f) = (1/\pi) \int_{-\infty}^{\infty} U(f')(f-f')^{-1} df'.$$ [2]

Here, $*$ denotes a convolution integral, and $\int$ designates the principle value of the
integral. This is a consequence of the causality principle which states that the impulse
response $h(t)$, which is the Fourier transform of the transfer function $Y(f)$, is zero for
negative time argument,

$$h(t) = 0 \text{ for } t < 0,$$ [3]

for any physical system.

In Fourier spectroscopy, the absorption mode spectrum $V(f)$ and the dispersion
mode spectrum $U(f)$ are computed from the experimentally measured impulse
response $h(t)$. Clearly, this general principle must be applicable here as well. Some
consequences of practical importance will be discussed in this paper.

In practical Fourier spectroscopy a periodic pulse sequence is used to excite the spin
system for reasons of sensitivity. It is shown in the section on periodic pulse excitation
that, in general, the computed absorption and dispersion mode signals are no longer
related. But, based on a modified impulse response supplemented by a period of equal
length but with zero amplitude, it is possible to obtain absorption and dispersion mode
spectra which are related by a discrete Hilbert transform (the latter was introduced in
Ref. (*4*)).

Practical realizations of Fourier spectroscopy require a discrete representation of the
impulse response by a finite number of sample values. Here, the discrete periodic

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Hilbert transform must be applied. In many applications of Fourier spectroscopy, it has been found to be convenient to increase the frequency resolution of the computed spectra by Fourier-transforming an impulse response supplemented by a much larger period with zero amplitude. It is shown in the section on interpolation of Fourier transforms that this process is equivalent to a trigonometric interpolation process which may improve the presentation but does not add to the available information.

Finally, the Hilbert transform relationship allows, in certain cases, improvement of the sensitivity by a complete utilization of the inherent information, as is shown in the final section. It will be assumed in this paper that the impulse response is measured directly as it is done in nuclear magnetic resonance Fourier spectroscopy (5, 6). On the other hand, the same principles apply as well to optical Fourier spectroscopy where the calculation of the spectrum is based on the interferogram which measures the auto-correlation function of the transmitted radiation or the crosscorrelation function between input and transmitted radiation (7–9).

In optical applications of Fourier spectroscopy, the assumption of linearity of the investigated system is an excellent approximation. But it is well known that nonlinear effects are important in nuclear magnetic resonance. Fortunately, they do not affect the response to a single strong radio frequency pulse, except for a scaling factor which depends in a nonlinear fashion on the strength of the exciting rf pulse, and the Fourier transform relationship still holds up to a scaling factor.

On the other hand, one may ask whether it is still possible to treat the nuclear spin system as a linear one when it is excited by a periodic sequence of rather strong rf pulses. In fact, real and imaginary parts of the Fourier transform of the periodic impulse response are in general not identical to the low power absorption and dispersion mode spectra of the investigated system, especially if the pulse spacing $T < T_2$ (as, for example, in Figs. 2 and 3). In practical applications of Fourier spectroscopy, however, it is desirable to interpret the Fourier transform of this periodic response as an approximate absorption and dispersion mode spectrum. Even in cases where this approximation is rather bad (e.g. when strong echo effects occur), the impulse response and its Fourier transform may still be useful for the detection of weak signals (14, 15).

From a general point of view, spectroscopy is system identification. This process consists of selecting a mathematical model based on the assumed structure of the system and of measuring the unknown parameters of this model. For a linear model, this parameter identification implies measuring the transfer function (complex spectrum) or the impulse response which is the Fourier transform of it. On the other hand, as soon as we attempt to Fourier analyze the impulse response of an arbitrary system we implicitly have accepted to approximate this system by a linear model.

The remaining question is: How appropriate is the selected linear model for the particular physical system? But this question, which will not be answered in this paper, does not affect the presented results which are correct whenever the impulse response of a physical system is Fourier transformed, irrespective of possibly nonlinear behavior of this system.

HILBERT TRANSFORM RELATIONSHIP FOR PERIODIC PULSE EXCITATION

The inherent gain of sensitivity of Fourier spectroscopy can only be realized when the system is periodically excited by means of a repetitive pulse sequence (5, 6). The

response will be periodic as well, and all information, except for the contributions of the random noise, is contained in a single period. For a periodic excitation, the computed absorption and dispersion mode spectra are, in general, independent. This can be shown as follows.

A periodic function $h^T(t)$ with the period length T contains only harmonics of the fundamental frequency $1/T$. Its Fourier transform is thus represented by an infinite but discrete set of Fourier coefficients

$$\mathscr{F}\{h^T(t)\} = \{Y_l^T\} = \{V_l^T + iU_l^T\}, \quad -\infty < l < \infty, \tag{4}$$

with

$$Y_l^T = \int_0^T h^T(t)\, e^{-i2\pi l t/T}\, dt. \tag{5}$$

It can easily be shown that $\{V_l^T\}$ is the Fourier transform of the even part $h_g^T(t)$ of $h^T(t)$:

$$h_g^T(t) = \tfrac{1}{2}[h^T(t) + h^T(-t)], \tag{6}$$
$$\mathscr{F}\{h_g^T(t)\} = \{V_l^T\}.$$

On the other hand, $i\{U_l^T\}$ is the Fourier transform of the odd part $h_u^T(t)$ of $h^T(t)$:

$$h_u^T(t) = \tfrac{1}{2}[h^T(t) - h^T(-t)], \tag{7}$$
$$\mathscr{F}\{h_u^T(t)\} = i\{U_l^T\}.$$

It is obvious that in general $h_g^T(t)$ and $h_u^T(t)$ are independent, and, therefore, there cannot exist a relation between the signals $\{V_l^T\}$ and $\{U_l^T\}$.

Because of the periodicity of the excitation, the causality principle does not apply in its common form, Eq. [2]. Real and imaginary parts of the Fourier transform of the periodic response are independent. It is now possible to enforce an artificial causality principle which then relates the computed absorption and dispersion mode spectrum. To this aim, it is necessary to supplement the periodic response function $h^T(t)$, defined in the interval $0 \leqslant t < T$, by the function values for a negative time interval of equal length (10):

$$h^{2T}(t) = \begin{cases} h^T(t) & \text{for} \quad 0 \leqslant t < T \\ 0 & \text{for} -T \leqslant t < 0 \end{cases}. \tag{8}$$

The function $h^{2T}(t)$ remains a periodic function but with the period $2T$.

The even and odd parts of $h^{2T}(t)$,

$$h_g^{2T}(t) = \tfrac{1}{2}[h^{2T}(t) + h^{2T}(-t)]$$

and

$$h_u^{2T}(t) = \tfrac{1}{2}[h^{2T}(t) - h^{2T}(-t)], \tag{9}$$

are now connected by the relations

$$h_u^{2T}(t) = h_g^{2T}(t)\, \text{sgn}\,([t - 2T]\,\text{mod}\,2T),$$
$$h_g^{2T}(t) = h_u^{2T}(t)\, \text{sgn}\,([t - 2T]\,\text{mod}\,2T). \tag{10}$$

The relations between $h^T(t)$, $h^{2T}(t)$, $h_g^{2T}(t)$, and $h_u^{2T}(t)$ are visualized in Fig. 1. Equation [10] implies that the real and imaginary part of the Fourier transform of $h^{2T}(t)$, $\{V_l^{2T}\}$ and $\{U_l^{2T}\}$, must be related by convolution sum of the form

$$\{U_l^{2T}\} = (-i/2T)\{V_l^{2T}\} * \mathscr{F}\{\text{sgn}\,([t - 2T]\,\text{mod}\,2T)\}. \tag{11}$$

The Fourier transform of $(i/2T)\{\text{sgn}\,([t - 2T]\,\text{mod}\,2T)\}$, $\{S_l\}$, is given by

$$S_l = (2/\pi)\sin^2(\pi l/2)/l, \quad l = \ldots - 2, -1, 0, 1, 2, \ldots \qquad [12]$$

or explicitly

$$\{S_l\} = (2/\pi)\{\ldots 0, -1/3, 0, -1, 0, 1, 0, 1/3, 0 \ldots\}. \qquad [12a]$$

This is the kernel of a convolution sum which relates U_l and V_l

$$\{U_l^{2T}\} = -\{V_l^{2T}\} * \{S_l\} = - \sum_{j=-\infty}^{\infty} V_j^{2T} S_{l-j},$$

$$\{V_l^{2T}\} = \{U_l^{2T}\} * \{S_l\} \quad = \sum_{j=-\infty}^{\infty} U_j^{2T} S_{l-j}. \qquad [13]$$

It was introduced in Ref. (4). This transformation is called the *discrete Hilbert transform*.

Absorption and dispersion mode signals obtained in a periodic Fourier experiment are discrete Hilbert transforms of each other whenever causality is enforced by supplementing the free induction decay signal by a period of equal length but with zero amplitude. Fourier transformation of this modified time signal produces twice as many Fourier coefficients as the transformation of the original signal. The new values lie centered between the original Fourier coefficients. Thus, this process is equivalent to an interpolation process. It increases the information content of either absorption or dispersion mode signal by the amount of additional information contained in the other signal. Both signals now contain the same information. It is interesting to note that absorption and dispersion mode signals now consist of two independent sets of interlaced Fourier coefficients. The even numbered coefficients of the absorption mode signals form the discrete Hilbert transform of the odd numbered coefficients of the dispersion mode signals and vice versa. This is caused by the particular structure of $\{S_l\}$. All the even numbered coefficients S_l are zero.
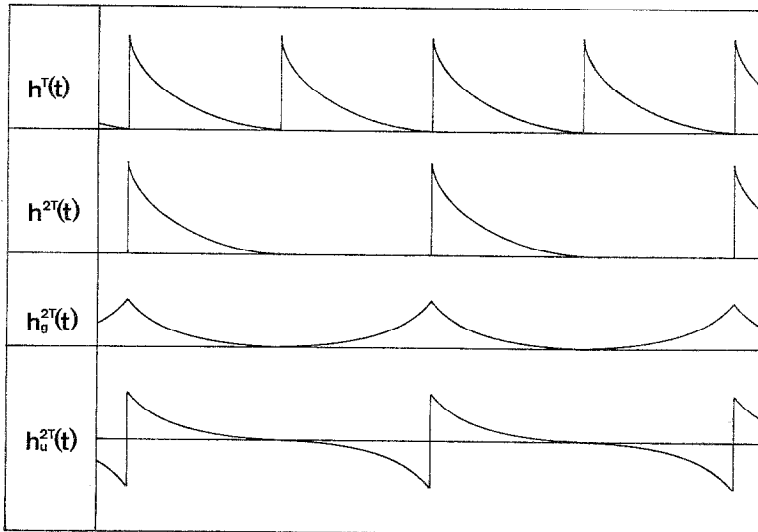

FIG. 1. The original periodic response $h^T(t)$, the modified periodic response $h^{2T}(t)$ and its even and odd part $h_g^{2T}(t)$, $h_u^{2T}(t)$.

HILBERT TRANSFORM RELATIONSHIP BASED ON A DISCRETE
REPRESENTATION OF THE IMPULSE RESPONSE

For the numerical Fourier transformation, it is necessary to represent the experimental impulse response by a discrete, finite sequence of N sample values $\{h_k\}^N$, $k = 0, 1, \ldots,$ $N - 1$, which are taken at equally spaced instances. In the course of the sampling process, information is lost necessarily unless the signal is strictly band-limited and the number of sample values is taken according to the sampling theorem (3). This information loss may be significant in practical applications but it will not be considered further in this paper.

The Fourier transform of a discrete, periodic time sequence is discrete and periodic itself and is represented by the same number N of sample values:

$$Y_l = \sum_{k=0}^{N-1} h_k e^{-i2\pi kl/N}, \quad l = 0, 1, \ldots, N - 1. \tag{14}$$

Because the sample values h_k are real, the relation $Y_l = Y_{N-l}$ holds. It implies that there are only $N/2$ independent complex Fourier coefficients. Real and imaginary parts of these coefficients are independent, in general, and the absorption and dispersion mode signals are unrelated. To enforce artificial causality and a Hilbert transform relationship between absorption and dispersion mode signal, it is again necessary to supplement the original sequence $\{h_k\}^N$ by N zeros to represent the continuation on the negative time axis (10):

$$\{h_k\}^{2N} = \{0, \ldots \ldots 0, h_0, h_1, \ldots \ldots h_{N-1}\}. \tag{15}$$

Because of the periodicity, it is immaterial whether the zeros are added to the left or to the right of the original sample values.

In complete analogy to the preceding section, it is possible to separate even and odd part of $\{h_k\}^{2N}$. To obtain a unique relation between them, however, it is additionally necessary to suppress h_0 and put it equal to zero. The value h_0 will, finally, give the integral of the absorption mode signal. This value cannot be obtained by a finite Hilbert transform starting from the dispersion mode signal.

Thus,

$$\{h_k\}_g^{2N} = \tfrac{1}{2}\{0, h_{N-1}, h_{N-2}, \ldots, h_1, 0, h_1, \ldots, h_{N-1}\} \tag{16}$$

and

$$\{h_k\}_u^{2N} = \tfrac{1}{2}\{0, -h_{N-1}, \ldots, -h_1, 0, h_1, \ldots \ldots h_{N-1}\}$$

with the relation

$$\{h_k\}_u^{2N} = \{\operatorname{sgn}(k) h_{kg}\}^{2N}. \tag{17}$$

The absorption mode signal $\{V_l\}^{2N}$ is the Fourier transform of the symmetric part $\{h_k\}_g^{2N}$ and the dispersion mode signal $i\{U_l\}^{2N}$ is the Fourier transform of the anti-symmetric $\{h_k\}_u^{2N}$. This gives finally the relations

$$\{U_l\}^{2N} = -\{V_l\}^{2N} * \{S_l\}^{2N} = - \sum_{j=0}^{2N-1} V_j S_{l-j},$$

$$\{V_l\}^{2N} = \{U_l\}^{2N} * \{S_l\}^{2N} = \sum_{j=0}^{2N-1} U_j S_{l-j}. \tag{18}$$

with the kernel of this transformation

$$S_l = (1/N) \sin^2 (\pi l/2) \cot (\pi l/2N). \tag{19}$$

This transformation may be called the *discrete periodic Hilbert transform* (11). The functional form of this transformation is very similar to the discrete Hilbert transform, Eq. [12]. There are again two independent sets of Fourier coefficients in each of the two signals. $\{Y_l\}^N$ and $\{Y_l\}^{2N}$ are related by $V_l^N = V_{2l}^{2N}$ and $U_l^N = U_{2l}^{2N}$.

In the case of a discrete representation of the impulse response, it is particularly instructive to follow the distribution of information among the two signals on the occasion of the Fourier transformation. In a Fourier transformation without using the mentioned interpolation procedure, the N degrees of freedom contained in the N real sample values of the impulse response are transformed into $N/2 + 1$ independent Fourier coefficients contained in the absorption mode signal, $V_0, V_1, \ldots V_{N/2}$, and into $N/2 - 1$ independent Fourier coefficients of the dispersion mode signal, $U_1, U_2, \ldots U_{N/2-1}$. The values U_0 and $U_{N/2}$ are necessarily zero and the coefficients with higher index values are related to the listed coefficients. It is clear that neither $\{V_l\}^N$ nor $\{U_l\}^N$ represent the complete information content of the impulse response $\{h_k\}^N$, and it is obvious that $\{V_l\}^N$ and $\{U_l\}^N$ are independent.

The Fourier transformation of the zero-supplemented impulse response, however, produces N independent Fourier coefficients for the absorption mode signal, $V_0, V_1, V_2, \ldots V_{N-1}$. It is quite obvious why the dispersion mode must contain one degree of freedom less than the absorption mode signal. The dispersion mode signal and the kernel function of the discrete periodic Hilbert transform are both antisymmetric functions with zero integral. This implies that the convolution sum $\{S_l\}^{2N}*\{U_l\}^{2N}$ must also have zero integral. The integral of the absorption mode signal which must be different from zero is this additional missing degree of freedom. It can never be obtained through a discrete periodic Hilbert transformation.

FURTHER INTERPOLATION OF FOURIER TRANSFORMS

In some cases, it may be desirable to compute further intermediate values of the absorption or dispersion mode spectrum to increase the frequency resolution. This can easily be achieved by a Fourier transformation of the original set of N sample values of the impulse response supplemented by $(2n - 1)N, n > 1$, virtual samples of zero amplitude (5). It is clear that this process can not further increase information of either signal because each signal already contains all of the information gained by the particular experiment. This process is strictly an interpolation procedure which may be desirable for a smooth representation of the spectrum or to provide a convenient measurement of accurate peak positions.

It will now be shown that this process is entirely equivalent to a simple trigonometric interpolation procedure (13). This may be seen as follows: The Fourier coefficient $Y(f)$ for an arbitrary frequency f is obtained in this procedure by the evaluation of the expression

$$Y(f) = \sum_{k=-N+1}^{N-1} h_k e^{-i2\pi fk/2N}. \qquad [20]$$

The extension of the lower limit of the summation in Eq. [20] is essential to obtain a real interpolation function, although the values of h_k, $k < 0$, are all zero. h_k can now be expressed by the discrete Fourier coefficients $\{Y_l\}^{2N}$,

$$Y(f) = \sum_{k=-N+1}^{N-1} \frac{1}{2N} \sum_{l=0}^{2N-1} Y_l e^{i2\pi lk/2N} e^{-i2\pi fk/2N}$$

$$= \sum_{l=0}^{2N-1} Y_l \frac{1}{2N} \sum_{k=-N+1}^{N-1} e^{-i2\pi(f-l)k/2N} = \sum_{l=0}^{2N-1} Y_l I(f-l). \qquad [21]$$

This expression may be interpreted as the result of a convolution integral of a discrete function formed by the Fourier coefficients Y_l and of a continuous interpolation function of the form

$$I(f) = \frac{1}{2N} \sum_{k=-N+1}^{N-1} e^{-i2\pi fk/2N}$$
$$= (1/N) \cos((N-1)\pi f/2N) \sin(\pi f/2)/\sin(\pi f/2N) + 1/2N. \qquad [22]$$

This is the trigonometric interpolation function for a periodic sequence of sample values. Interpolation must be extended only over one period. For $N \to \infty$, we obtain the well known interpolation function for an aperiodic sequence of sample values,

$$\lim_{N\to\infty} I(f) = \sin(\pi f)/\pi f = \text{sinc}(f). \qquad [23]$$

This shows clearly that the interpolated values $Y(f)$ are simple functions of the significant Fourier coefficients $\{Y_l\}^{2N}$ and do not provide additional information. It is important to notice that the interpolation function $I(f)$ is real. This implies that the interpolation process does not mix absorption and dispersion mode signals. Interpolation processes of the real or of the imaginary part of the transfer function are independent. On the other hand, when interpolation is based on the original Fourier coefficients $\{Y_l\}^N$, the corresponding interpolation function would be complex, because here interpolation requires the information contained in the quadrature component, as well. The interpolation function for $\{Y_l\}^N$ is given by

$$I(f) = (1/N)(1 - e^{-i2\pi f})(1 - e^{-i2\pi f/N})^{-1}, \qquad [22a]$$

and $Y(f)$, by

$$Y(f) = \sum_{l=0}^{N-1} Y_l I(f-l). \qquad [21a]$$

When a fast Fourier transform algorithm is used for the computation of n interpolated values between adjacent original Fourier coefficients, nN sample values, including $(n-1)N$ zeros, must be transformed. The number n is thereby limited by the capacity of the computer memory. For this reason and to minimize computing time, it may in some cases be more convenient to apply direct interpolation to $\{V_l\}^{2N}$ or $\{U_l\}^{2N}$ by means of Eq. [21] or to $\{Y_l\}^N$ by means of Eq. [21a]. Here, the summation may be limited to a few terms to speed up computation. It is also possible to employ conventional polynomial interpolation techniques (12, 13) to interpolate $\{V_l\}^{2N}$ or $\{U_l\}^{2N}$.

When interpolation is effected by means of a zero-supplemented impulse response, it is important to first subtract the mean amplitude from all original sample values to avoid a discontinuity in the modified impulse response. This would cause strong oscillations in the transformed spectrum.

EXAMPLE OF AN INTERPOLATED FOURIER TRANSFORM

A rather extreme example will be considered to clarify some of the results presented in the preceding sections. It is assumed that the impulse response of a nuclear spin system in an inhomogeneous static magnetic field is recorded. The inhomogeneity shall cause a line broadening by a factor 100 such that $1/T_2^* = 100/T_2$. It is further

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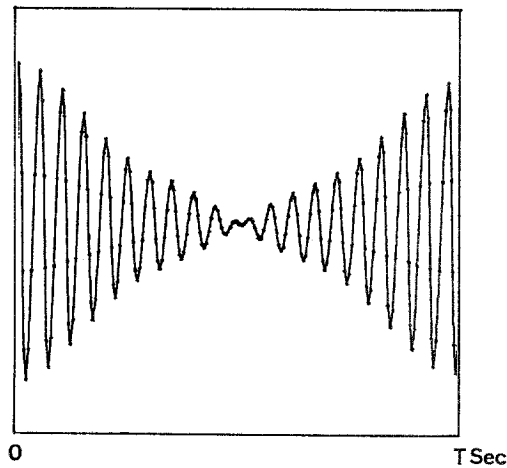0   T Sec

FIG. 2. Computed impulse response signal for a magnetic resonance system obeying Bloch's equations with $T_2/T_2^* = 100$, $T_1/T_2 = 1$, $T/T_2^* = 3$, flip angle $\alpha = 75°$ and for a resonance line coinciding with a harmonic of the pulse sequence.
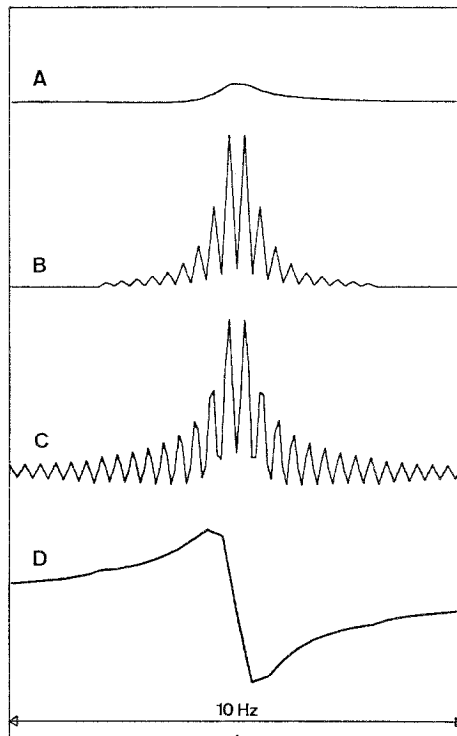

FIG. 3. Fourier transform of the impulse response of Fig. 2. (A) Cosine transform of the original impulse response. (B) Cosine transform of the impulse response supplemented by N values of zero amplitude. (C) Cosine transform of the impulse response supplemented by $3N$ values of zero amplitude. (D) Sine transform of the original impulse response.

assumed that $T_1 = T_2$ and that the flip angle employed is $\alpha = 75°$. The pulse separation T is selected to be $T = 3T_2^*$, and the resonance is supposed to coincide with one of the harmonics of the pulsing sequence. The obtained response, represented by 100 sample values, is shown in Fig. 2. It is characterized by the occurrence of a very strong echo or refocusing effect immediately before the next applied rf pulse (14). This is clearly an artifact of the measuring technique, i.e., caused by the close pulse spacing, $T \ll T_2$.

The corresponding absorption mode spectrum without interpolation computed from this free induction decay is shown in Fig. 3A. Absorption is rather weak because the
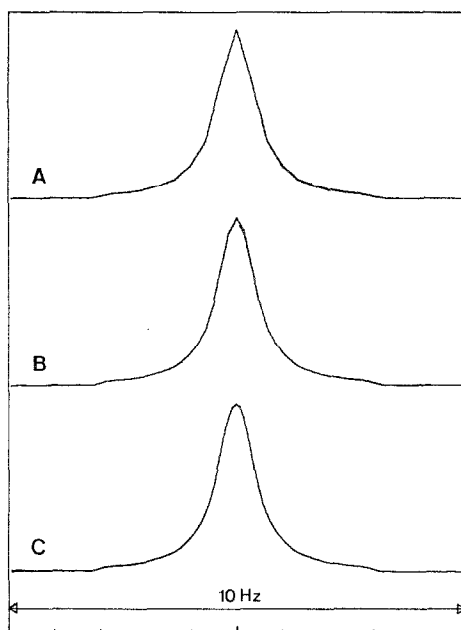

FIG. 4. Cosine-Fourier transform of the backfolded impulse response of Fig. 2 (compare Eq. [24]). (A) Fourier transform of the sequence of $N/2$ values. (B) Fourier transform of the sequence supplemented by $N/2$ values of zero amplitude. (C) Fourier transform of the sequence supplemented by $3N/2$ values of zero amplitude.

original impulse response is almost antisymmetric and its symmetric part has very low amplitude. The addition of N zero values to the impulse response and the cosine-Fourier transform of the $2N$ samples produces the absorption mode spectrum of Fig. 3B. The additional interpolated values represent the discrete periodic Hilbert transform of the original dispersion mode spectrum (Fig. 3D) which has a comparatively very high amplitude due to the particular antisymmetric shape of the impulse response. This causes the extreme oscillation in Fig. 3B. This trace contains now all the information of the N values of the recorded impulse response. Further interpolation, by addition of another $2N$ zero values to the impulse response, does not drastically change the absorption mode signal anymore and does not add information (Fig. 3C).

The undesirable intensity oscillation shown in the interpolated absorption mode

spectra can be avoided by Fourier transformation of a modified sequence obtained from the impulse response $\{h_k\}^N$:

$$\{h_m\}^{N/2} = \{h_0, h_1 - h_{N-1}, \ldots, h_{N/2-1} - h_{N/2+1}\} \quad \text{(for even } N\text{)}, \qquad [24]$$

which is obtained by subtracting the inverted second half of the sequence from the first half. The envelope of this modified sequence goes smoothly to zero towards its upper end. The results of the Fourier transform of this modified sequence are shown in Fig. 4A without interpolation, in Fig. 4B with single interpolation, and in Fig. 4C with double interpolation. It is seen that in this case no drastic changes occur due to interpolation. Some slight line broadening must be put up with, but it is considerably weaker than any conventional filtering process would produce in an attempt to filter out the oscillations shown in Figs. 3B and 3C. A smooth, almost Lorentzian shape is obtained finally.

SENSITIVITY ENHANCEMENT

The Fourier transformation of the zero-supplemented impulse response $\{h_k\}^{2N}$ yields a gain in information compared with the Fourier transform of the original impulse response $\{h_k\}^N$. This gain can also be interpreted in terms of a sensitivity improvement.

The maximum achievable signal-to-noise ratio of a signal of known line shape superimposed by random noise of known spectral density is obtained by means of a matched filter. For white random noise, it can be shown that the maximum signal-to-noise ratio achievable per unit time is given by (6)

$$(S/N)^2_{\max} = (1/T) \sum_l |V_l|^2/W, \qquad [25]$$

where W is the power spectral density of the random noise which is constant over the whole frequency range.

In the case of the Fourier transform of the original impulse response $\{h_k\}^N$, the summation in Eq. [25] runs over $N/2$ independent Fourier coefficients. On the other hand, for the zero supplemented impulse response, the summation has to be extended over N Fourier coefficients. Thus, the maximum possible sensitivity enhancement is given by

$$\left(\frac{(S/N)_{\max,2N}}{(S/N)_{\max,N}}\right)^2 = \sum_{l=0}^{N-1} |V_l^{2N}|^2 \Big/ \sum_{l=0}^{N/2-1} |V_l^N|^2. \qquad [26]$$

This holds because the power spectral density of the random noise is the same in both cases. The N Fourier coefficients of the random noise remain uncorrelated during this interpolation process. It may seem possible to even further enhance sensitivity by further interpolation as described previously. This is actually impossible because the further Fourier coefficients are correlated. At maximum N Fourier coefficients are independent. Thus, after the first interpolation, no further gain is possible.

For an impulse response signal which decays almost to zero during the pulse spacing T, e.g. for $T > 3T_2$, symmetric and antisymmetric part are essentially equal in signal energy. Thus the signal energy of the sequence $\{V_l\}^{2N}$ has about twice the signal energy of the sequence $\{V_l\}^N$. In this case, the sensitivity improvement is about a factor $\sqrt{2}$.

If, however, the impulse response signal is to a high degree antisymmetric as for example in Fig. 2, the enhancement of the inherent sensitivity by the first interpolation

in the absorption mode signal is much more striking as can be seen by comparing Fig. 3A with Fig. 3B.

The gain in sensitivity by central interpolation relies on the fact that after interpolation the number of samples within a line width is higher such that more filtering can be tolerated without excessive signal distortion. Filtering is effective only when the Fourier coefficients of the random noise are uncorrelated. As soon as apodization of the impulse response is employed for enhancement of sensitivity, this gain is partially anticipated and the Fourier coefficients of the random noise become partially correlated.

This also implies that for the case that no interpolation is employed filtering in the time domain by apodization is more effective than filtering in the final spectrum by means of a convolution process. In the case of an interpolated signal, the two filtering procedures are equivalent.

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